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Heat and Harmonic Extensions for Function Spaces of Hardy–Sobolev–Besov Type on Symmetric Spaces and Lie Groups

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Function spaces of Hardy–Sobolev–Besov type on symmetric spaces of noncompact type and unimodular Lie groups are investigated. The spaces were originally defined by uniform localization. In the paper we give a characterization of the space $F^s_{p,q}(X)$ and $B^s_{p,q}(X)$ in terms of heat and Poisson semigroups, for $1 \leqslant p, q \leqslant \infty$ and $s \in \mathbb{R}$. The main tool we use, is an atomic decomposition of function spaces on manifolds. © 1999 Academic Press

1. PRELIMINARIES

Let (X, g) be an n-dimensional connected Riemannian manifold with the Riemannian metric tensor g. Let r_{inj} denote the injectivity radius of X. The manifold X is called a manifold of bounded geometry if the following two conditions are satisfied:

(a) $r_{inj} > 0$,

(b) $|\nabla^k R| \leqslant C_k$, $k = 0, 1, 2, ...$, (i.e., every covariant derivative of the Riemannian curvature tensor is bounded).

Examples of manifolds of bounded geometry include all compact manifolds and all homogeneous spaces, i.e., manifolds with a transitive group of isometries (symmetric spaces, Lie groups with left (right) Riemannian structure).

Let $\{\Omega(x_j, r)\}_j$ be a covering of X by geodesic balls. The maximal number of the balls with non-empty intersection in this covering is called the multiplicity of the covering. A covering with finite multiplicity is called uniformly locally finite. For the manifold X of bounded geometry there exists a number $0 < r_0 < r_{inj}$ such that if $r \in (0, r_0)$ then there exists a countable uniformly locally finite covering of X by balls of radius r, cf.

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[10, Lemma 1.2]. Furthermore, for every uniformly locally finite covering $\{\Omega(x_j, r)\}_j$ there exists a corresponding resolution of unity $\{\varphi_i\} \subset C_o^\infty(X)$ with the following properties:

$$0 \leqslant \varphi_j \leqslant 1, \qquad \operatorname{supp} \varphi_j \subset \Omega(x_j, r), \qquad j = 1, 2, ..., \qquad \sum_j \varphi_j = 1 \qquad \text{on } X; \tag{1}$$

for any multi-index α there exists a positive number b_α with

$$|D^\alpha(\varphi \circ \exp_{x_j})| \leqslant b_\alpha, \qquad j = 1, 2, ... \tag{2}$$

We assume that the reader is familiar with the definition and elementary properties of function spaces of $F^s_{p,q} - B^s_{p,q}$ type on $\mathbb{R}^n$. All we need can be found in [22]. To define the $F^s_{p,q}$ spaces on the manifold X one can use the localization property of the space $F^s_{p,q}(\mathbb{R}^n)$ and their invariance with respect to a wide class of diffeomorphisms.

Definition 1 (cf. [20]). Let $\{\varphi_j\}$ be the above resolution of unity.

1. Let either $0 < p < \infty$, $0 < q \leqslant \infty$, or $p = q = \infty$. Let $-\infty < s < \infty$. Then

$$F^s_{p,q}(X) = \left\{ f \in \mathscr{D}'(X) \colon \|f \mid F^s_{p,q}(X)\| = \left(\sum_j \|\varphi_j f \circ \exp_{x_j} \mid F^s_{p,q}(\mathbb{R}^n)\|^p \right)^{1/p} < \infty \right\}$$

(with the usual modification if $p = \infty$).

2. Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. Let $-\infty < s_0 < s < s_1 < \infty$. Then

$$B^s_{p,q}(X) = (F^{s_0}_{p,p}(X), F^{s_1}_{p,p}(X))_{\theta, q}$$

with $s = (1-\theta)\, s_0 + \theta s_1$.

Remark 1. 1. The definition is independent of the chosen resolution of unity and the numbers s_0, s_1 in the Besov case.

2. The function spaces on manifolds have a lot of properties analogous to the space $F^s_{p,q}(\mathbb{R}^n) - B^s_{p,q}(\mathbb{R}^n)$, cf. [18, 20, 22]. In particular we have

$$\begin{array}{lll} F^0_{p,2} = L_p & (1 < p < \infty) & \text{Lebesgue spaces,} \\ F^s_{p,2} = W^s_p & (1 < p < \infty,\ s \in \mathbb{N}) & \text{Sobolev spaces,} \\ F^s_{p,2} = H^s_p & (1 < p < \infty,\ s \in \mathbb{R}) & \text{Bessel potential spaces,} \\ B^s_{\infty,\infty} = \mathscr{C}^s & (s > 0) & \text{Höler–Zygmund spaces,} \end{array} \tag{3}$$

where the Sobolev space W^s_p is defined in terms of covariant derivatives and H^s_p is a Bessel potential space for the Laplace–Beltrami operator Δ of X.

In the paper we use widely the atomic decomposition of the spaces $F^s_{p,q}(X) - B^s_{p,q}(X)$, so we recall the construction. The details can be found in [11]. Let r_j, $j=0, 1, 2, \ldots$ be a sequence of positive numbers decreasing to zero. Let $(\Omega_j = \{\Omega(x_{j,i}, r_j)\}_{i=0}^{\infty})_{j=0}^{\infty}$ be a sequence of uniformly locally finite coverings of X. The supremum of multiplicities of coverings Ω_j, $j=0, 1, \ldots$, is called the multiplicity of the sequence Ω_j. The sequence Ω_j is called uniformly finite if its multiplicity is finite and the balls $\Omega(x_{j,i}, r_j/2)$ and $\Omega(x_{j,k}, r_j/2)$ have empty intersection for any possible j, i, k, $k \neq i$.

Lemma 1 (cf. [11]). *There exist $r_0 > 0$ such that for every $r \in (0, r_0)$. There is a uniformly finite sequence (Ω_j) of coverings of X by geodesic balls of radius $r_j = 2^{-j}r$, $\Omega_j = \{\Omega(x_{j,i}, r_j)\}_{i \in \mathbb{N}}$, $j=0, 1, \ldots$. Moreover, if $l \in \mathbb{N}$ and $l \cdot r < r_0$ then the multiplicity of the sequence $(\Omega_j^{(l)})_{j=0,1,\ldots}$, $\Omega_j^{(l)} = \{\Omega(x_{j,i}, lr_j)\}_{i \in \mathbb{N}}$, is also finite.*

Definition 2 (cf. [11]). Let $s \in \mathbb{R}$ and $0 < p \leqslant \infty$. Let L and M be integers such that $L \geqslant 0$ and $M \geqslant -1$. Let $r > 0$ and $C \geqslant 1$ be constants such that $Cr < \frac{1}{2} r_{inj}$.

(a) A smooth function $a(x)$ is called an 1_L-atom centered in $\Omega(x, r)$ if

$$\operatorname{supp} a \subset \Omega(x, 2Cr), \tag{4}$$

$$\sup_{y \in X} |\nabla^k a(y)| \leqslant C \qquad \text{for any} \quad |k| \leqslant L. \tag{5}$$

(b) A smooth function $a(x)$ is called an $(s, p)_{L,M}$-atom centered in $B(x, r)$ if

$$\operatorname{supp} a \subset \Omega(x, 2Cr), \tag{6}$$

$$\sup_{y \in X} |\nabla^k a(y)| \leqslant C r^{s-k-(n/p)}, \qquad \text{for any} \quad k \leqslant L, \tag{7}$$

$$\left| \int_X a(y)\, \psi(y)\, dy \right| \leqslant C r^{s+M+1+n/p'} \, \| \psi \mid C^{M+1}(\bar{\Omega}(x, 2r)) \| \tag{8}$$

holds for any $\psi \in C_0^\infty(\Omega(x, 3r))$.

If $M = -1$ then (8) means that no moment conditions are required.

Definition 3. Let $\Omega_j = \{\Omega(x_{j,i}, r_j)\}_{i \in \mathbb{N}}$, $j=0, 1, \ldots$, be a uniformly finite sequence of coverings. Let $s \in \mathbb{R}$ and $0 < p \leqslant \infty$. Let L and M

be integers satisfying the assumption of Definition 2. A family $\mathscr{A}^{L,M}_{s,p}$ if 1_L-atoms and $(s,p)_{L,M}$-atoms is called a building family of atoms corresponding to the sequence $\{B_j\}$ if:

(a) all atoms belonging to the family are centered at the balls of the coverings Ω_j;

(b) all atoms belonging to the family satisfy the conditions (4)–(8) with the same positive constant C;

(c) the family contains all atoms satisfying (a)–(b).

For $c \in \mathbb{R}$ let $[c]$ stand for the largest integer less than or equal to c and $C_+ = \max(c, 0)$. Moreover, for the characteristic function $\chi_{j,i}$ of the ball $\Omega(x_{j,i}, 2^{-j})$ we put $\chi^{(p)}_{j,i} = 2^{jn/p}\chi_{j,i}$.

THEOREM 1 (cf. [11]). *Let $s \in \mathbb{R}$, $0 < q \leqslant \infty$. Let $0 < p < \infty$ or $p = q = \infty$ in the case of the $F^s_{p,q}$-scale and $0 < p \leqslant \infty$ in the case of $B^s_{p,q}$-scale. Let L and M be fixed integers satisfying the following condition*

$$L \geqslant ([s]+1)_+ \qquad \textit{and}$$

$$M \geqslant \max\left(\left[n\left(\frac{1}{\min(p,q)}-1\right)_+ - s\right], -1\right). \tag{9}$$

There exist a positive constant ε_0, $0 < \varepsilon_0 \leqslant r_0$ such that there is a uniformly sequence of covering $\{\Omega_j = \{\Omega(x_{j,i}, r_j)\}_{i\in\mathbb{N}}\}$, $r < \varepsilon_0$, and a building family of atoms corresponding to the sequence $\mathscr{A}^{L,M}_{s,p}$ with the following properties:

(a) *each $f \in F^s_{p,q}(X)$ ($f \in B^s_{p,q}(X)$) can be decomposed as follows*

$$f = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} s_{j,i}a_{j,i} \qquad (\textit{convergent in } \mathscr{D}'(X)) \tag{10}$$

$$\left\|\left(\sum_{j,i=0}^{\infty}(|s_{j,i}|\,\chi^{(p)}_{j,i}(\cdot))^q\right)^{1/q}\right\|_p < \infty,$$

$$\left(\left(\sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty}|s_{j,i}|^p\right)^{q/p}\right)^{1/q} < \infty\right). \tag{11}$$

(b) *Conversely, suppose that $f \in \mathscr{D}'(X)$ can be represented as in* (10) *and* (11). *Then $f \in F^s_{p,q}(X)$ ($f \in B^s_{p,q}(X)$).*

Furthermore, the infimum of (11) *with respect to all admissible representations (for fixed sequence of coverings and fixed integers L, M) is an equivalent norm in $F^s_{p,q}(X)$ ($B^s_{p,q}(X)$).*

The heat kernel on Riemannian manifolds was a subject of intensive study during the last decades, cf. [3]. The heat semi-group $H_t = e^{t\Delta}$ can be defined on $L_2(X)$ by the spectral theorem and then extended to a positivity-preserving contraction semi-group on L_p for $1 \leqslant p \leqslant \infty$. The conversation of probability condition, $\forall t\ e^{t\Delta} 1 = 1$, holds and

$$\frac{\partial}{\partial t} e^{t\Delta} f = \Delta e^{t\Delta} \quad \forall f \in L_p, \qquad 1 \leqslant p \leqslant \infty.$$

The semi-group is strongly continuous if $1 \leqslant p < \infty$ and analytical if $1 < p < \infty$.

The exist a heat kernel $k_t(x, y)$ that is a strictly positive C^∞-function on $(0, \infty) \times X \times X$, symmetric in the space variables such that

$$c^{t\Delta} f = \int_X k_t(x, y) f(y)\, dy, \qquad f \in L_p(X), \qquad 1 \leqslant p \leqslant \infty$$

where dy denotes the Riemannian measure on X.

We consider also the Poisson semigroup $P_t = e^{-t\sqrt{-\Delta}}$ which can be obtained from H_t by the subordination formula

$$P_t = \frac{1}{\sqrt{\pi}} \int_0^\infty u^{-1/2} e^{-u} H_{t^2/4u}\, du, \qquad t > 0. \tag{12}$$

2. FUNCTION SPACES ON SYMMETRIC SPACES

Let $X = G/K$ be a Riemannian symmetric space of the noncompact type. We use the standard notation and refer to [6] and [7] for more details. In particular we have the Iwasawa decomposition $G = KAN$ of the group G and its Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. We call that the L_p-Schwartz spaces $\mathscr{C}_p(X)$ $(0 < p \leqslant 2)$ on X are defined as follows:

$$\mathscr{C}_p(X) = \Big\{ f \in C^\infty(X) \colon \sup_{\substack{k_1, k_2 \in K \\ H \in \mathfrak{a}}} (1 + |H|^2)^{r/2}\, \Xi^{-2/p}$$
$$\times (\exp H)\, |f(D_1 \colon k_1(\exp H)\, k_2 \colon D_2)| < \infty,\ D_1, D_2 \in U(\mathfrak{g}),\ r \geqslant 0 \Big\},$$

where Ξ is an elementary spherical function and $f(D_1 \colon k_1(\exp H)\, k_2 \colon D_2)$ denote the natural action of $D_1, D_2 \in U(\mathfrak{g})$ (the universal enveloping algebra of $\mathfrak{g}$) on $f \in C^\infty(G)$, cf. [4]. Their are Frechet spaces and $C_o^\infty(X) \subset \mathscr{C}_p(X) \subset L_q(X)$ if $p \leqslant q$ (but not for $p > q$). The dual spaces $\mathscr{C}'_p(X)$ are spaces of distributions on X.

Let Mf denote the Hardy–Littlewood maximal operator on X and $M_o f$ the local Hardy–Littlewood operator. We will need the following version of the Fefferman–Stein maximal inequality.

LEMMA 2 (cf. [14]). *Let $1 < p < \infty$ and $1 < q \leqslant \infty$. Then*

$$\left\| \left(\sum_{j=1}^{\infty} |Mf_j(\cdot)|^q \right)^{1/q} \right\|_p \leqslant C_{p,q} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q \right)^{1/q} \right\|_p \tag{13}$$

holds for any sequence of locally integrable functions on X.

2.1. *Heat Extension Characterization*

The heat semigroup $H_t = e^{t\Delta}$ $(t > 0)$ on $X = G/K$ is realized by convolution on the right with the heat kernel h_t. The kernel h_t is a positive bi-K-invariant L_1-Schwartz function on X for any $t > 0$. We have a good pointwise estimate for the heat kernel due to J.-Ph. Anker cf. [1]. In the paper we use mainly the estimates for $0 < t < t_o$, so we recall them here. We put

$$h_t^m(x) = \left(\frac{\partial}{\partial t} \right)^m h_t(x); \qquad m = 0, 1, ... \tag{14}$$

let $0 < t < t_o$ and $H \in \overline{\mathfrak{a}_+}$. Then there is constant C depending on t_o such that the inequality

$$|h_t^m(\exp H)| \leqslant C e^{-|\rho|^2 t - \rho(H) - |H|^2/4t} t^{-n/2 - 2m} \langle J \rangle^{n-\alpha} \sum_{l=0}^{m} t^l \, |H|^{2m-2l} \tag{15}$$

holds for every $m = 0, 1, 2, ...$, $\langle H \rangle = (1 + |H|^2)^{1/2}$, cf. [1].

The following standard observation is crucial for the paper. The heat semigroup is analytic in $L_2(X)$, therefore $\|t^k (d^k/dt^k) H_t\|_{2 \to 2} \leqslant C$. Moreover, H_t is a bounded operator from L_2 into L_∞ and $\|H_t\|_{2,\infty} \leqslant t^{-\nu}$, $\nu > 0$, cf. [2]. Using the last inequalities is not hard to see that

$$t^l \left(\frac{d}{dt} \right)^l h_t * f(x) \to 0 \qquad \text{if} \quad t \to 0$$

for any $f \in \mathscr{C}_1(X)$ and every $x \in X$.

Integrating by parts we get

$$\int_0^1 t^k \left(\frac{d^k}{dt^k} H_t f \right) \frac{dt}{t} = \sum_{l=1}^{k-1} c_l h_1^l * f(x) - cf(x)$$

Thus

$$f(x) = C\left(h_{m,0} * f + \int_0^1 t^k \frac{d^k}{dt^k} H_t f \frac{dt}{t}\right) \tag{16}$$

if $f \in \mathscr{C}_1(X)$ where $h_{k,0} = \sum_{l=0}^{k-1} c_l h_1^l$.

Moreover, if $t \to t_o$, $0 < t_o \leqslant 1$ then

$$t^l \left(\frac{d}{dt}\right)^l h_t * f \to t_o^l \left(\frac{d}{dt}\right)^l h_{t_o} * f$$

in $\mathscr{C}_1(X)$. In consequence (16) is true for every $f \in \mathscr{C}'_1(X)$ if the convergence of the integral is understood in weak sense.

THEOREM 2. *Let $s \in \mathbb{R}$, $1 \leqslant q \leqslant \infty$ and $m > |s|/2$.*

(i) *Let $1 \leqslant p < \infty$ or $p = q = \infty$. Then*

$$\|f \mid F^s_{p,q}(X)\|^{(m)}_H = \|f * h_{0,m}\|_p + \left\| \left(\int_0^1 t^{(m-s/2)q} \left| \frac{d^m}{dt^m} H_t f(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \right\|_p \tag{17}$$

is an equivalent norm in $F^s_{p,q}(X)$. Furthermore

$$F^s_{p,q}(X) = \{ f \in \mathscr{C}'_1(X) \colon \|f \mid F^s_{p,q}(X)\|^{(m)}_H < \infty \}. \tag{18}$$

(ii) *Let $1 \leqslant p \leqslant \infty$. Then*

$$\|f \mid B^s_{p,q}(X)\|^{(m)}_H = \|f * h_{0,m}\|_p + \left(\int_0^1 t^{(m-s/2)q} \left\| \frac{d^m}{dt^m} H_g f \right\|_p^q \frac{dt}{t} \right)^{1/q} \tag{19}$$

is an equivalent norm in $B^s_{p,q}(X)$. Furthermore

$$B^s_{p,q}(X) = \{ f \in \mathscr{C}'_1(X) \colon \|f \mid B^s_{p,q}(X)\|^{(m)}_H < \infty \}. \tag{20}$$

If $s > 0$ then in both cases $\|f * h_{0,m}\|_p$ can be replaced by $\|f\|_p$.

Proof. We focus our attention on $F^s_{p,q}(X)$ spaces. The proof for the Besov spaces is similar but easier, for example we do not need the maximal inequalities for vector-valued function.

Step 1. We take $t_o = 1$. Then the inequality (15) implies

$$|t^m h_t^m(\exp H)| \leqslant C t^{-n/2} e^{-(1/4)(|H|/\sqrt{t})^2}, \qquad \text{for} \quad |H| \leqslant \sqrt{t}, \tag{21}$$

$$|t^m h_t^m(\exp H)| \leqslant C t^{-n/2} \left(\frac{|H|}{\sqrt{t}}\right)^2 m e^{-(1/4)(|H|/\sqrt{t})^2}, \qquad \text{for} \quad \sqrt{t} \leqslant |H| \leqslant 1, \tag{22}$$

$$|t^m h_t^m(\exp H)| \leqslant C e^{-\rho(H) - |H|^2/8} \langle H \rangle^{n-\alpha}, \qquad \text{for} \quad |H| \geqslant 1. \tag{23}$$

Moreover the function on the right hand side of (23) defines a bi-K-invariant integrable function on X in the usual way. For the proof we refer to [12].

It will be convenient to introduce the following spaces

$$\mathscr{F}^{s,m}_{p,q}(X) = \{ f \in \mathscr{C}'(X) \colon \| f \mid F^s_{p,q}(X) \|^{(m)}_H < \infty \}, \tag{24}$$

$$\mathscr{B}^{s,m}_{p,q}(X) = \{ f \in \mathscr{C}'(X) \colon \| f \mid B^s_{p,q}(X) \|^{(m)}_H < \infty \}. \tag{25}$$

Step 2. For further use we need two inequalities for maximal function.

Let Φ be a non-negative radial function defined on X supported in $\Omega(o, 1)$. There is a positive constant C such that the inequality

$$|\Phi * f(x)| \leqslant C \int_X \Phi(y)\, dy (M_o |f|)(x) \tag{26}$$

holds for any locally integrable function f. The proof of the above inequality is standard. It is sufficient to prove (26) for Φ normalized by $\int \Phi\, dx = 1$. First take Φ of the form $\sum_{l=1}^{\infty} a_l \chi_{\Omega(o, r_l)}$, where each a_j is positive. Then, since $\sum_l a_l \operatorname{vol}(\Omega(o, r_l)) = 1$ and $\chi_{\Omega(o, r_1)} * |f(x)| \leqslant C \operatorname{vol}(\Omega(o, r_l))(M_o |f|)(x)$ the inequality (26) follows immediately. In general the function Φ can be approximated by such finite sums, so (26) holds as claimed.

Let $\varepsilon > 1$ and $\tilde{\chi}_{j,i}$ denote the characteristic function of the ball $\Omega(x_{j,i}, \varepsilon 2^{-j})$. Moreover we put $\tilde{\chi}^{(p)}_{j,i} = 2^{j_n/p} \tilde{\chi}_{j,i}$. Then the following elementary inequality

$$M_o(\chi^{(p)}_{j,i})(x) \leqslant C(M(\tilde{\chi}^{(p)\,w}_{j,i}))^{1/w}(x), \tag{27}$$

holds for any $0 < w < 1$ with the constant C independent of j.

Step 3. We assume that $s>0$. Let

$$f=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} s_{j,i}a_{j,i} \qquad \text{with} \quad \left\| \left(\sum_{j,i=0}^{\infty} (|s_{j,i}|\,\chi_{j,i}^{(p)}(\cdot))^q \right)^{1/q} \right\|_p < \infty.$$

Then

$$\|f\,|\,F^s_{p,q}(X)\|_H^{(m)} \leqslant \left\| \sum_{j,i=0}^{\infty} s_{j,i}h_{0,m} * a_{j,i} \right\|_p + \left\| \left(\int_0^1 t^{(m-s/2)\,q} \left| \sum_{j,i=0}^{\infty} s_{j,i}h_t^m * a_{j,i}(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \right\|_p .$$

We estimate every summand separately.

It should be clear that putting $\tilde{\chi}_{j,i}^{(p)}$ instead of $\chi_{j,i}^{(p)}$ in (11) we get an equivalent norm. This observation, the integrability of $h_{0,m}$ and the definition of the atoms give us

$$\left\| \sum_{j,i=0}^{\infty} s_{j,i}h_{0,m} * a_{j,i} \right\|_p \leqslant C \left\| \left(\sum_{j,i=0}^{\infty} |s_{j,i}|^q\, \tilde{\chi}_{j,i}^{(p)}(\cdot)^q \right)^{1/q} \right\|_p . \tag{28}$$

To estimate the second summand we first note that the inequalities (21)–(23) and (26) imply

$$\begin{aligned}
\int_X t^m\, |h_t^m|\,(y) \left(\sum_{i=0}^{\infty} |s_{j,i}|\, \chi_{j,i}^{(p)}(y^{-1}x) \right) dy
&= \int_{|y|\leqslant\sqrt{t}} + \int_{\sqrt{t}\,|y|\leqslant 1} + \int_{|y|\geqslant 1} \\
&\leqslant CM_o \left(\sum_{i=0}^{\infty} |s_{j,i}|\, \chi_{j,i}^{(p)} \right)(x) + Ch * \left(\sum_{i=0}^{\infty} |s_{j,i}|\, \chi_{j,i}^{(p)} \right)(x)
\end{aligned}$$

where h is a non-negative integrable function on X. We divide the second summand into two parts,

$$\begin{aligned}
&\left(\int_0^1 t^{(m-s/2)\,q} \left| \sum_{j,i=0}^{\infty} s_{j,i}h_t^m * a_{j,i}(x) \right|^q \frac{dt}{t} \right)^{1/q} \\
&\quad\leqslant \left(\sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} t^{(m-s/2)\,q} \left(\sum_{j=0}^{[k/2]} \left| \sum_{i=0}^{\infty} s_{j,i}h_t^m * a_{j,i}(x) \right| \right)^q \frac{dt}{t} \right)^{1/q} \qquad (29) \\
&\qquad + \left(\sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} t^{(m-s/2)\,q} \left(\sum_{j=[k/2]}^{\infty} \left| \sum_{i=0}^{\infty} s_{j,i}h_t^m * a_{j,i}(x) \right| \right)^q \frac{dt}{t} \right)^{1/q} . \qquad (30)
\end{aligned}$$

We put $J = \min(m, [L/2])$. If $j \leqslant [k/2]$ then $(2j-k)(2J-s)$ is a nonpositive number. Thus, by the definition of atoms the sum (29) is less or equal to

$$\Bigg(\sum_{k=0}^{\infty}\Bigg(\sum_{j=0}^{[k/2]} \sqrt{2}^{(2j-k)(2J-s)} \sup_{0<t\leqslant 1} t^{m-J}\,|h_t^{m-J}| * \sum_{i=0}^{\infty} |s_{j,i}|\,\tilde{\chi}^{p}_{j,i}(x)\Bigg)^q\Bigg)^{1/q}$$
$$\leqslant C\Bigg(\sum_{j=0}^{\infty}\Bigg(M\Bigg(\sum_{i=0}^{\infty}|s_{j,i}|\,\tilde{\chi}^{(p)}_{j,i}\Bigg)(x)\Bigg)^q\Bigg)^{1/q}$$
$$+\,C\Bigg(\sum_{j=0}^{\infty}\Bigg(h * \sum_{i=0}^{\infty}|s_{j,i}|\,\tilde{\chi}^{p}_{j,i}(x)\Bigg)^q\Bigg)^{1/q} \tag{31}$$

Moreover, (27) and the elementary inequality $M(f)^q \leqslant M(f^q)$ imply that the following inequality

$$\sum_{j=0}^{\infty}\Bigg(M\Bigg(\sum_{i=0}^{\infty}|s_{j,i}|\,\chi^{(p)}_{j,i}\Bigg)(x)\Bigg)^q \leqslant C\sum_{j,\,i=0}^{\infty}(M(|s_{j,i}|^w\,\tilde{\chi}^{(p)\,w}_{j,i})(x))^{q/w} \tag{32}$$

holds.

Let $0 < w < 1$ be such that $\min(q/w, p/w) > 1$. Then the Fefferman–Stein maximal inequality, the Minkowski inequality for integrals and (31)–(32) imply

$$\Bigg\|\Bigg(\sum_{k=0}^{\infty}\int_{2^{-k-1}}^{2^k} t^{(m-s/2)\,q}\Bigg(\sum_{j=0}^{[k/2]}\Bigg|\sum_{i=0}^{\infty} s_{j,i}h_t^m * a_{j,i}(\cdot)\Bigg|\Bigg)^q\frac{dt}{t}\Bigg)^{1/q}\Bigg\|$$
$$\leqslant C\Bigg\|\Bigg(\sum_{i,\,j=0}^{\infty}|s_{j,i}|^q\,\tilde{\chi}^{(p)}_{j,i}(\cdot)^q\Bigg)^{1/q}\Bigg\|_p. \tag{33}$$

In the similar way we get the estimate for the sum (30). Now $k-2j\leqslant 0$. So,

$$\Bigg\|\Bigg(\sum_{k=0}^{\infty}\int_{2^{-k-1}}^{2^k} t^{(m-s/2)\,q}\Bigg(\sum_{j=[k/2]}^{\infty}\Bigg|\sum_{i=0}^{\infty} s_{j,i}h_t^m * a_{j,i}(\cdot)\Bigg|\Bigg)^q\frac{dt}{t}\Bigg)^{1/q}\Bigg\|_p$$
$$\leqslant C\Bigg\|\Bigg(\sum_{k=0}^{\infty}\Bigg(\sum_{j=[k/2]}^{\infty}\sqrt{2}^{(k-2j)\,s}\sup_{0<t\leqslant t} t^m\,|h_t^m| * \sum_{i=0}^{\infty}|s_{j,i}|\,\tilde{\chi}^{(p)}_{j,i}(\cdot)\Bigg)^q\Bigg)^{1/q}\Bigg\|_p$$
$$\leqslant C\Bigg\|\Bigg(\sum_{j,\,i=0}^{\infty}|s_{j,i}|\,\tilde{\chi}^{(p)}_{j,i}(\cdot)^q\Bigg)^{1/q}\Bigg\|_p.$$

Thus we have proved for $s > 0$ the following inequality

$$\|f\,|\,F^s_{p,q}(X)\|^{(m)}_H \leqslant C\Bigg\|\Bigg(\sum_{j,\,i}|s_{j,i}|^q\,\chi^{(p)}_{j,i}(\cdot)^q\Bigg)^{1/q}\Bigg\|_p. \tag{34}$$

Step 4. To deal with decompositions of distributions from $\mathcal{F}^{s,m}_{p,q}(X)$ into atoms we need some inequalities. Let us take a uniformly locally finite sequence $\{\Omega_j\}$ of coverings of X with $r_j = \varepsilon 2^{-j}$, where ε is a fixed number such that $0 < \varepsilon < 1$.

For future use we need two positive constants b and δ. We choose these constants in such a way that the following identities are satisfied $b - \delta^2 = b/16$ and $b - \delta = b/4$. Such constants exist and both b and δ are greater then 1. Let $Q_{j,i} = (b4^{-j-1}, b4^{-j}) \times \Omega(x_{j,i}, 2^{-j})$. Then the Harnack–Moser inequality for subsolution of parabolic equations implies

$$\sup_{(t,x) \in Q_{j,i}} |h^m_t * f(x)|$$

$$\leqslant C2^{jn/w} \left(\int_{\Omega(x_{j,i}, \delta 2^{-j})} \int_{b4^{-j-2}}^{b4^{-j}} |h^m_t * f(x)|^w \frac{dt}{t} dx \right)^{1/w} \tag{35}$$

where C is the constant depending only on n, b, δ and w, $0 < w < \infty$, cf. Theorem 5.5 in [9].

For reasons that will be clear later on we assume that $\varepsilon b > 1$. Let $\{\psi_{j,i}\}$ be the smooth resolution of unity corresponding to the covering $\{\Omega(x_{j,i}, \varepsilon 2^{-j})\}$. We may assume that for every positive m there is a constant b_m such that the inequality

$$\left| \frac{\partial^{|\gamma|}}{\partial H^\gamma} \psi_{j,i} \circ \exp_{x_{j,i}}(H) \right| \leqslant b_m 2^{-j|\gamma|} \tag{36}$$

holds for every j, i and every $H \in T_{x_{j,i}} X$ and every multi-index γ such that $|\gamma| \leqslant m$. The Theorem III.1.5 in [24] the scaling method, cf. Section V.3 ibidem, imply that the inequality

$$|\nabla^k h^m_t * f(x)| \leqslant C2^{j(k+n)} \int_{Q_{j,i}} |h^m_t * f(y)| \frac{dt}{t} dy \tag{37}$$

holds for any $(t, x) \in [\varepsilon b 4^{-j-1}, \varepsilon b 4^{-j}] \times \Omega(x_{j,i}, \varepsilon 2^{-j})$.

Now we decompose any distribution from $\mathcal{F}^{s,m}_{p,q}(X)$, $s > 0$, into atoms. We start with the formula (16). Since $\mathscr{C}_p(X)$ is dense in $\mathscr{C}'_p(X)$ the formula (16) if true for any $f \in \mathscr{C}'_p(X)$ provided that the convergence in (16) is understood in the weak $\mathscr{C}'_p(X)$ sense. For this part of proof it is convenient to change the formula (16) a bit and to rewrite it down in the following form

$$f(x) = C\left(h_{m,0} * f + \int_0^{\varepsilon b} t^k \left(\frac{d^k}{dt^k} H_t f \right) \frac{dt}{t} \right), \tag{38}$$

where b is the positive constant from the last step. Since $h_{m,0}=\sum_{l=0}^{m-1} h^l_{\varepsilon b}$ and $\varepsilon b>1$ we can write

$$h_{m,0} * f=\sum_{l=0}^{m-1} h^l_{\varepsilon b-1} * h_1 * f. \tag{39}$$

Let $\{E_i\}$ be a decomposition of X into a sum of disjoint sets such that $E_i \subset \Omega(x_{0i}, \varepsilon)$. Let $GE_i=\pi^{-1}(E_i)$, $\pi\colon G \mapsto X$ is a natural projection.

Using the above resolutions of unity and (38)–(39) we get the following decomposition of f

$$\begin{aligned} f(x) &= C\left(h_{m,0} * f+\int_0^{\varepsilon b} t^k \left(\frac{d^k}{dt^k} H_t f\right) \frac{dt}{t}\right) \\ &= C\left(h_{m,0} * f+\sum_{j=1,\, i=0}^{\infty} \psi_{j,i} \int_{\varepsilon b 4^{-j-1}}^{\varepsilon b 2^{-j}} t^m h^m_t * f\frac{dt}{t}\right) \\ &= C\left(\sum_{j,\, i=0}^{\infty} s_{j,i} a_{j,i}\right) \end{aligned}$$

where

$$a_{j,i}(x)=2^{-2jm} s_{j,i}^{-1} \psi_{j,i}(x) \int_{\varepsilon b 4^{-j-1}}^{\varepsilon b 4^{-j}} t^m h^m_t * f(x) \frac{dt}{t} \qquad \text{for} \quad j \geqslant 1 \tag{40}$$

$$a_{0,i}(x)=s_i^{-1} \psi_{0,i}(x) \int_{GE_i} f * h_1(g) \left(\sum_{l=0}^{l-1} h^l_{\varepsilon b-1}(g^{-1}x)\right) dg, \tag{41}$$

$$s_{j,i}=2^{j(s-(n/p)-2m)} \sum_{l \in I_i} \sup_{x \in Q_{j,l}} |h_t * \Delta^m f|\,(x) \qquad \text{for} \quad j \geqslant 1 \tag{42}$$

$$s_{0,i}=\left(\int_{\Omega(x_{0,i},\, 1)} |f * h_1(x)|^p\, dx\right)^{1/p}, \tag{43}$$

and

$$I_i=\{l \in \mathbb{N} : \Omega(x_{j,l}, 2^{-j}) \cap \Omega(x_{j,i}, 2^{-j}) \neq \varnothing\}.$$

It follows from inequalities proved in this step that $a_{j,i}$ are (s, p)-atoms cf. (35)–(37). The functions $a_{0,i}$ are 1_L atoms because $h^l_{\varepsilon b-1}$ are L_1-Schwartz functions.

Step 5. It should be clear that the expression

$$\|f\|_p + \left\| \left(\sum_{j=1}^{\infty} \int_{b4^{-j-2}}^{b4^{-j}} t^{(m-s/2)q} \left| \left(\frac{\partial}{\partial t}\right)^m f * h_t \right|^q (\cdot) \frac{dt}{t} \right)^{1/q} \right\|_p$$

is an equivalent norm in $\mathscr{F}^{s,m}_{p,q}(X)$ if $s > 0$. We use that expression to estimate the atomic norm from above. Once more we choose w such that $\min(q/w, p/w) > 1$. Using the Fefferman–Stein maximal inequality [11] we get

$$\left\| \left(\sum_{j=0}^{\infty} \int_{b4^{-j-2}}^{b4^{-j}} t^{(m-s/2)q} \left| \left(\frac{\partial}{\partial t}\right)^m f * h_t \right|^q (\cdot) \frac{dt}{t} \right)^{1/q} \right\|_p$$

$$\geqslant C \left\| \left(\sum_{j=0}^{\infty} M \left(\int_{b4^{-j-2}}^{b4^{-j}} t^{(m-s/2)q} \left| \left(\frac{\partial}{\partial t}\right)^m f * h_t \right|^w \frac{dt}{t} \right)^{q/w} (\cdot) \right)^{1/q} \right\|_p$$

$$\geqslant C \left\| \left(\sum_{j,i=0}^{\infty} 2^{jq(s-(n/p))} 4^{-jmq} \right.\right.$$

$$\left.\left. \times M \left(\int_{b4^{-j-2}}^{bb^{-j}} \left| \left(\frac{\partial}{\partial t}\right)^m f * h_t \right|^w \frac{dt}{t} \right)^{q/w} (\cdot) \, \chi^{(p)}_{j,i}(\cdot)^q \right)^{1/q} \right\|_p .$$

But there is a constant C independent of j and i such that the inequalities

$$\left(M \left(\int_{b4^{-j-2}}^{b4^{-j}} \left| \left(\frac{\partial}{\partial t}\right)^m f * h_t \right|^w \frac{dt}{t} \right) \right)^{1/w} (x)$$

$$\geqslant C \sum_{l \in I_i} (\delta 2)^{nj/w} \cdot \left(\int_{\Omega(x_{i,l}, \delta 2^{-j})} \int_{b4^{-j-2}}^{b4^{-j}} \left| \left(\frac{\partial}{\partial t}\right)^m f * h_t \right|^w (y) \frac{dt}{t} \, dy \right)^{1/w}$$

$$\geqslant C \sum_{l \in I_i} \sup_{x \in Q_{j,l}} |h_t * \Delta^m f| \, (x)$$

holds for any $l \in I_i$. Therefore

$$\left\| \left(\sum_{j=0}^{\infty} \int_{b4^{-j-2}}^{b4^{-j}} t^{(m-s/2)q} \left| \left(\frac{\partial}{\partial t}\right)^m f * h_t \right|^q (\cdot) \frac{dt}{t} \right)^{1/q} \right\|_p \tag{44}$$

$$\geqslant C \left\| \left(\sum_{j,i=o}^{\infty} |s_{j,i}|^q \, \chi^{(p)}_{j,i}(\cdot)^q \right)^{1/q} \right\|_p . \tag{45}$$

This proves the theorem for $s > 0$.

Step 6. It remain to prove the theorem for $s \leqslant 0$. The operator $(I - \Delta)^{-1}$ maps the space $\mathscr{C}_1(X)$ into $\mathscr{C}_1(X)$. Thus it can be extended to $\mathscr{C}'_1(X)$. We prove that if $2k > -s$ then the operator $(I - \Delta)^{-k}$ defines the isomorphism of the space $\mathscr{F}^{k,s}_{p,q}(X)$ ($\mathscr{B}^{k,s}_{p,q}(X)$) onto $\mathscr{F}^{s+2k}_{p,q}(X) = F^{s+2k}_{p,q}(X)$ ($\mathscr{B}^{s+2k}_{p,q}(X) = B^{s+2k}_{p,q}(X)$). Since it is known that the operator $(I-\Delta)^{-k}$ defines the isomorphism of the space $F^s_{p,q}(X)$ ($B^s_{p,q}(X)$) onto $F^{s+2k}_{p,q}(X)$ ($B^{s+2k}_{p,q}(X)$) the last fact will finish the proof of the theorem.

Let $f \in \mathscr{F}^{s+2k}_{p,q}(X)$. If $s<0$ then $k>k+s/2$. So,

$$\|(I-\Delta)^k f * h_{0,k}\|_p + \left\| \left(\int_0^1 t^{(k-s/2)q} \, |h^k_t * (I-\Delta)^k f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_p$$

$$\leqslant C \|f\|_p + \sum_{l=0}^{k} C \left\| \left(\int_0^1 t^{(k+l-(s+2k)/2)q} \, |h^{k+l}_t * f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_p$$

If $s<0$ then $k+l>k+(s/2)$ for any possible l so every summand in the last sum is less or equaled to $\|f \mid F^{s+2k}_{p,q}\|$. If $s=0$ then $k+l>k+(s/2)$ for $l=1, ..., k$ so there is only the small problem with the first summand for which we have

$$\left\| \left(\int_0^1 t^{tq} \, |h^k_t * f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_p \leqslant \left\| \left(\int_0^1 t^{(k-\sigma/2)q} \, |h^k_t * f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_p$$

$$\leqslant \|f \mid F^{s+2k}_{p,q}\|$$

where $0<\sigma<k$. In consequence

$$\|(I-\Delta)^k f * h_{0,k} \mid F^s_{p,q}(X)\|^{(k)}_H \leqslant C \, \|f \mid F^{s+2k}_{p,q}(X)\|. \tag{46}$$

In the similar way one can prove the analogous inequality for Besov spaces.

The operator $(-\Delta)^k (I-\Delta)^{-k}$ in $L_p(X)$, $1 \leqslant p \leqslant \infty$. Using the last fact it is not hard to see that $(I-\Delta)^{-k}$ maps $\mathscr{B}^{k,s}_{p,q}(X)$ onto $B^{s+2k}_{p,q}(X)$. This finish the proof for Besov spaces.

Now we assume that $f \in \mathscr{F}^{k,s}_{p,q}(X)$. Using the method due to E. Stein, one can prove by spectral theory that

$$(-\Delta)^k (I-\Delta)^{-k} h^k_t * f = h^k_t * f + \sum c_m (I-\Delta)^{-m} h^k_t * f,$$

with $\sum |c_m| < \infty$, cf. [15, p. 133]. Since $2k > (s+2k)/2$ we have

$$\left\| \left(\int_0^1 t^{(2k-(s+2k)/2)q} \, |h^{2k}_t * (I-\Delta)^{-k} f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_p$$

$$\leqslant C \, \|f \mid F^s_{p,q}\|^{(k)}_H$$

$$+ C \left\| \left(\int_0^1 t^{(k-s/2)q} \left| \sum c_m h^k_t * (I-\Delta)^{-m} f(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \right\|_p.$$

Using the Hölder and Minkowski inequality for integrals one can prove easily that $B^{s_1}_{p,q}(X) \subset \mathscr{F}^{s,k}_{p,q}(X) \subset B^{s_0}_{p,p}(X)$, if $s-1 < s_0 < s < s_1 < s+1$. But the operators $(I-\Delta)^{-m}$ are bounded from $B^{s_1}_{p,p}(X)$ to $B^{s_1}_{p,p}(X)$ therefore the last inequalities imply

$$\left\| \left(\int_0^1 t^{(2k-(s+2k)/2)\,q}\, |h^{2k}_t * (I-\Delta)^{-k} f(\cdot)|^q \frac{dt}{t} \right)^{1/q} \right\|_p$$

$$\leqslant C\, \|f \,|\, F^s_{p,q}\|^{(k)}_H + \left\| \sum c_m (I-\Delta)^{-m} f \,|\, B^{s_1}_{p,p}(X) \right\|$$

$$\leqslant C\, \|f \,|\, F^s_{p,q}\|^{(k)}_H + \sum |c_m|\, \|f \,|\, B^{s_0}_{p,p}(X)\|$$

$$\leqslant C\, \|f \,|\, F^s_{p,q}\|^{(k)}_H .$$

The definition of $h_{2k,0}$ implies $h_{2k,0} = h_{k,0} + \Delta^k h_{k,0}$. Thus

$$\|(I-\Delta)^{-k} f \,|\, F^{s+2k}_{p,q}\|^{(k)}_H \leqslant \|f \,|\, F^s_{p,q}\|^{(k)}_H . \qquad (47) \ ∎$$

2.2. *Harmonic Extension Characterization*

In this section we consider the Poisson semigroup $P_t = e^{-t(-\Delta)^{1/2}}$. On X the semigroup P_t is realized by convolution on the right with the Poisson kernel p_t, which is a positive bi-K-invariant Schwartz function. Thus similar arguments as in Section 2.1 give us the following formula

$$f(x) = C\left(p_{m,0} * f + \int_0^1 t^k \frac{d^k}{dt^k} P_t f \frac{dt}{t} \right) \qquad (48)$$

if $f \in \mathscr{C}'_1(X)$ where $p_{m,0} = \sum_{l=0}^{m-1} c_l p_1^l$.

THEOREM 3. *Let $s \in \mathbb{R}$, $1 \leqslant q \leqslant \infty$ and $m > |s|$.*

(i) *Let $1 \leqslant p < \infty$ or $p = q = \infty$. Then*

$$\|f \,|\, F^s_{p,q}(X)\|^{(m)}_P = \|f * p_{0,m}\|_p + \left\| \left(\int_0^1 t^{(m-s)\,q} \left| \frac{d^m}{dt^m} P_t f(\cdot) \right|^q \frac{dt}{t} \right)^{1/q} \right\|_p \qquad (49)$$

is an equivalent norm in $F^s_{p,q}(X)$. Furthermore

$$F^s_{p,q}(X) = \{ f \in \mathscr{C}'_1(X) \colon \|f \,|\, F^s_{p,q}(X)\|^{(m)}_P < \infty \}. \qquad (50)$$

(ii) *Let $1 \leqslant p \leqslant \infty$. Then*

$$\|f \,|\, B^s_{p,q}(X)\|^{(m)}_P = \|f * p_{0,m}\|_p + \left(\int_0^1 t^{(m-s)\,q} \left\| \frac{d^m}{dt^m} P_t f \right\|_p^q \frac{dt}{t} \right)^{1/q} \qquad (51)$$

is an equivalent norm in $B^s_{p,q}(X)$. *Furthermore*

$$B^s_{p,q}(X) = \{ f \in \mathscr{C}'_1(X) \colon \| f \mid B^s_{p,q}(X) \|^{(m)}_P < \infty \}. \tag{52}$$

If $s > 0$ *then in both cases* $\| f * p_{0,m} \|_p$ *can be replaced by* $\| f \|_p$.

Proof. We divide the proof into several steps. Once more we concentrate on $F^s_{p,q}(X)$ spaces.

Step 1. We should prove the following inequalities. In this step we prove that the following inequalities

$$\| f \|_p + \left\| \left(\int_0^1 t^{(m-s)q} \, | p^m_t * f |^q (\cdot) \frac{dt}{t} \right)^{1/q} \right\|_p \leqslant C \, \| f \mid F^s_{p,q}(X) \| \tag{53}$$

hold for $s > 0$ and m even.

Let $m = 2k > s$. By the subordination formula we have

$$\begin{aligned} p_t(x) &= \frac{1}{2\sqrt{\pi}} \, t \int_0^\infty u^{-3/2} e^{-t^2/4u} h_u(x) \, du \\ &= \frac{1}{2\sqrt{\pi}} \, t \int_0^1 + \frac{1}{2\sqrt{\pi}} \, t \int_1^\infty . \end{aligned} \tag{54}$$

Let us fix $x \in X$. If $q = 1$ then

$$\int_0^1 t^{2k-s} \left| t \int_0^1 u^{-3/2} e^{-t^2/4u} h^k_u * f(x) \, du \right| \frac{dt}{t} \leqslant C \int_0^1 u^{m-s} \, | h^k_u * f(x) | \, \frac{du}{u}.$$

If $q = \infty$ then

$$\sup_{0 \leqslant t \leqslant 1} t^{2k-s} \left| t \int_0^1 u^{-3/2} e^{-t^2/4u} h^k_u * f(x) \, du \right| \leqslant C \sup_{0 \leqslant u \leqslant 1} u^{k-s/2} \, | h^k_u * f(x) |.$$

By interpolation we get

$$\begin{aligned} &\left\| \left(\int_0^1 t^{(2k-s)q} \left| t \int_0^1 u^{-3/2} e^{-t^2/4u} h^k_u * f(x) \, du \right|^q \frac{dt}{t} \right)^{1/q} \right\|_p \\ &\qquad \leqslant C \left\| \left(\int_0^1 u^{(k-s/2)q} \, | h^k_u * f(x) \, du |^q \, \frac{du}{u} \right)^{1/q} \right\|_p \end{aligned} \tag{55}$$

for any $1 \leqslant q \leqslant \infty$.

On the other hand if $u>1$ then $(d/du)^l h_u=(-\Delta)^l h_1 * h_{u-1}$. The heat semigroup is a contraction semigroup and $e^{t^2/4u}\leqslant 1$ for $u>1$, therefore the last identity gives us

$$\left\|\left(\int_0^1 t^{(2k-s)q}\left|t\int_1^\infty u^{-3/2}e^{-t^2/4u}h_u^k * f(\cdot)\,du\right|^q \frac{dt}{t}\right)^{1/q}\right\|$$

$$\leqslant \left\|\int_1^\infty u^{-3/2}(-\Delta)^k h_1 * h_{u-1} * f(\cdot)\,du\right\|_p$$

$$\leqslant C\int_1^\infty u^{-3/2}\,\|(-\Delta)^k h_1 * h_{u-1} * f(x)\|_p\,du\leqslant C\leqslant C\,\|f\|_p.$$

The last inequalities and (55) imply (53) for m even. The proof for Besov spaces is similar.

Step 2. Now we prove the inequality inverse to (53) for $s>0$ and any possible m. We use once more the atomic decomposition.

Let $\Omega_{j,i}=[2^{-j+1}, 2^{-j}]\times\Omega(x_{j,i}, 2^{-j})$. For the function $v(t,x)=p_t^m * f(x)$ we have $((\partial^2/\partial t^2)+\Delta)\,v=0$. So by the standard elliptic estimates we have

$$|\nabla^k p_t^m * f(x)|\leqslant C2^{jk}\sup_{(x,t)\in\Omega_{j,i}^\varepsilon}|p_t^m * f(x)| \tag{56}$$

where $\Omega_{j,i}^\varepsilon=[\varepsilon^{-1}2^{-j+1}, \varepsilon 2^{-j}]\times\Omega(x_{j,i}, \varepsilon 2^{-j})$, $\varepsilon>1$.

On the other hand we have the submean value property for subharmonic function, so the inequality

$$\sup_{\Omega_{j,i}}|p_t^m * f(x)|\leqslant C2^{-jn/w}\left(\int_{\Omega_{j,i}^\delta}|p_t^m * f(x)|^w\,dx\right)^{1/w} \tag{57}$$

holds for suitable $\delta>1$ and $0<w\leqslant 2$, cf. [8].

Using (48) we get the following decomposition of f

$$f(x)=C\left(\int_{i=0}^\infty \psi_{0,i}(x)\,p_{m,0} * f(x)+\sum_{j=1,\,i=0}^\infty \psi_{j,i}(x)\int_{2^{-j}}^{2^{-j+1}} t^m p_t^m * f(x)\frac{dt}{t}\right)$$

$$=C\left(\sum_{j,\,i=0}^\infty s_{j,i}a_{j,i}\right)$$

where

$$a_{j,i}(x) = s_{j,i}^{-1} \psi_{j,i}(x) \int_{2^{-j+1}}^{2^{-j}} t^m p_t^m * f(x) \frac{dt}{t} \qquad \text{for} \quad j \geqslant 1 \tag{58}$$

$$a_{0,i}(x) = s_i^{-1} \psi_{0,i}(x) \int_{GE_i} f * p_{1/2}(g) \left(\sum_{l=0}^{l-1} p_{1/2}^l(g^{-1}x) \right) dg, \tag{59}$$

$$s_{j,i} = 2^{j(s-(n/p)-m)} \sum_{l \in I_i} \sup_{x \in \Omega_{j,l}^{\varepsilon}} |p_t^m * f| (x) \qquad \text{for} \quad j \geqslant 1 \tag{60}$$

$$s_{0,i} = \left(\int_{\Omega(x_{0,i}, 1)} |f * p_{1/2}(x)|^p \, dx \, g \right)^{1/p}, \tag{61}$$

and

$$I_i = \{ l \in \mathbb{N} : \Omega(x_{j,l}, 2^{-j}) \cap \Omega(x_{j,i}, 2^{-j}) \neq \emptyset \}.$$

The rest of the proof is similar to the proof of Theorem 2.

Step 3. It follows from the above steps that the theorem is true for $s > 0$ and even m. Using the arguments similar to those in proof of Theorem 2 one can proved that if $s \leqslant 0$ and $s + 2k > 0$ then $(I - \Delta)^{-k}$ is an isomorphism of the space $\mathscr{F}_{p,q}^{s,m} = \{ f \in \mathscr{C}_1'(X) : \| f \mid F_{p,q}^s \|_P^{(m)} < \infty \}$ onto $F_{p,q}^{s+2k}(X)$ ($\mathscr{F}_{p,q}^{s,m}$ onto $B_{p,q}^{s+2k}(X)$).

It remains to regard the odd $m = 2k + 1$. The operator $(-\Delta)^{1/2}$ $(I - \Delta)^{-1/2}$ is bounded in $L_p(X)$ therefore for $s > 0$ we have

$$\| f \|_p + \left(\int_0^1 t^{(2k+1-s)q} \, \| p_t^{2k+1} * f \|^q \frac{dt}{t} \right)^{1/q}$$
$$\leqslant \| (I - \Delta)^{1/2} f \mid B_{p,q}^{s-1}(X) \| \leqslant C \, \| f \mid B_{p,q}^s(X) \|.$$

For the $F_{p,q}^s(X)$ spaces we can use once more Stein's representation of the operator $(-\Delta)^{1/2} (I - \Delta)^{-1/2}$, cf. *Step* 6 of the proof of Theorem 2. For a nonpositive s we can use the lift property as above. ▮

3. FUNCTION SPACES ON LIE GROUPS

Now we regard a connected unimodular Lie group G equipped with a left invariant Riemannian metric. In this case a Riemannian volume element coincides with the Haar measure on G and the Laplace–Beltrami operator is equaled to the sum of squares of left invariant vector fields

$$\Delta = \sum_{k=1}^{n} \tilde{X}_k^2, \tag{62}$$

where $X_1, ..., X_n$ is an orthonormal basis of $T_e G$. Thus we can use all results concerning heat semigroups generated by sum of squares on Lie groups.

The heat semigroup H_t is given by a right convolution:

$$H_t f(x) = \int_G f(y)\, h_t(y^{-1}x)\, dy \tag{63}$$

where $(t, x) \to h_t(x)$ is a C^∞ function on $\mathbb{R}_+ \times G$ and a positive solution of $((\partial/\partial t) + \Delta)u = 0$. The semigroup is once more symmetric submarkovian, hence analytic in $L_p(G)$ if $1 < p < \infty$. So, arguments similar to that ones which where used is the case of symmetric spaces give us the formula

$$f(x) = C\left(h_{m,0} * f + \int_0^1 t^k \frac{d^k}{dt^k} H_t f \frac{dt}{t}\right) \tag{64}$$

for $f \in C_o^\infty(X)$.

The Sobolev embeddings and Theorem II.4.2 in [24] implies that the uniform convergence $t^l(d/dt)^l h_t * f \to t_o^l(d/dt)^l h_{t_o} * f$ if $t \to t_o \leqslant 1$. So the formula (64) is true for any regular distribution f on G if the convergence of the integral is understood in the weak sense.

We have the following local version of Lemma 2.

LEMMA 3. *Let* $M_o f = \sup_{0<r<T} \operatorname{vol}(\Omega(e, r))^{-1} \chi_{\Omega(e,r)} * |f|$. *Let* $1 < p < \infty$ *and* $1 < q \leqslant \infty$. *Then the inequalities*

$$\operatorname{vol}\left\{x \in X: \left(\sum_{j=1}^\infty |M_o f_j(x)|^q\right)^{1/q} > \lambda\right\} \leqslant \frac{C_q}{\lambda} \left\|\left(\sum_{j=1}^\infty |f_j|^q\right)^{1/q}\right\|_1, \tag{65}$$

$$\left\|\left(\sum_{j=1}^\infty |M_o f_j(\cdot)|^q\right)^{1/q}\right\|_p \leqslant C_{p,q} \left\|\left(\sum_{j=1}^\infty |f_j|^q\right)^{1/q}\right\|_p \tag{66}$$

holds for any sequence of locally integrable functions on g.

The main result of this section reads as follow

THEOREM 4. *Let* $s \in \mathbb{R}$, $1 \leqslant q \leqslant \infty$, $m > |s|/2$ *and* $k > |s|$.

(i) *Let* $1 \leqslant p \leqslant \infty$ *or* $p = q = \infty$. *Then*

$$\|f \mid F^s_{p,q}(G)\|_H = \|f * h_{0,m}\|_p + \left\|\left(\int_0^1 t^{(m-s/2)q} \left|\frac{d^m}{dt^m} H_t f(\cdot)\right|^q \frac{dt}{t}\right)^{1/q}\right\|_p, \tag{67}$$

$$\|f \mid F^s_{p,q}(G)\|_P = \|f * p_{0,k}\|_p + \left\|\left(\int_0^1 t^{(k-s)q} \left|\frac{d^k}{dt^k} P_t f(\cdot)\right|^q \frac{dt}{t}\right)^{1/q}\right\|_p \tag{68}$$

are equivalent norms in $F^s_{p,q}(G)$. *Furthermore*

$$F^s_{p,q}(G)=\{f\in\mathscr{D}'(G)\colon \|f\mid F^s_{p,q}(G)\|_H<\infty\},$$
$$F^s_{p,q}(G)=\{f\in\mathscr{D}'(G)\colon \|f\mid F^s_{p,q}(G)\|_P<\infty\}.$$

(ii) *Let* $1\leqslant p\leqslant\infty$. *Then*

$$\|f\mid B^s_{p,q}(G)\|_H=\|f*h_{0,m}\|_p+\left(\int_0^1 t^{(m-s/2)\,q}\left\|\frac{d^m}{dt^m}H_tf\right\|_p^q\frac{dt}{t}\right)^{1/q} \tag{69}$$

$$\|f\mid B^s_{p,q}(G)\|_P=\|f*p_{0,k}\|_p+\left(\int_0^1 t^{(k-s)\,q}\left\|\frac{d^k}{dt^k}P_tf\right\|_p^q\frac{dt}{t}\right)^{1/q} \tag{70}$$

are equivalent norms in $B^s_{p,q}(G)$. *Furthermore*

$$B^s_{p,q}(G)=\{f\in\mathscr{D}'(G)\colon \|f\mid B^s_{p,q}(G)\|_H<\infty\},$$
$$B^s_{p,q}(G)=\{f\in\mathscr{D}'(G)\colon \|f\mid B^s_{p,q}(G)\|_P<\infty\}.$$

Proof. The above theorem can be prove in the similar way as Theorem 2 and Theorem 3, so we give only the main ideas of the proof.

Examining the proof of Theorem 2 we easily discover that the inequality

$$\|f\mid F^s_{p,q}(G)\|_H\leqslant\|f\mid F^s_{p,q}(G)\| \tag{71}$$

holds for $s>0$ if we have at our disposal the local version of the Fefferman–Stein inequalities and the pointwise estimates of the heat kernel analogous to the estimates (21)–(23). The Beltrami–Laplace operator coincided with the sum of squares of left invariant vector-fields therefore one can use the estimates for the sum of squares on Lie groups that are due to Varopoulos and others cf. [24]. Since the vectors $X_1, ..., X_n$ span the Lie algebra of G the Riemannian d distance on G equals to the distance induce by the system of vector fields $X_1, ..., X_n$.

Let $|x|=d(e,x)$. We have the following estimates for the heat kernel:

- there exists $c>0$ such that for all $t\in(0,1)$, for all $x\in G$,

$$h_t(x)\leqslant Ct^{-n/2}e^{-|x|^2/ct}, \tag{72}$$

cf. [24, Theorem V.4.2];

- for all $c\in(0,1)$, t_1, t_2 such that $0<t_1<t_2<\infty$ and $m\in\mathbb{N}$, there exists $C>0$ such that: $\forall x\in G$, $\forall s\in(0,1)$, $\forall u$ is a positive solution of $((\partial/\partial t)+\Delta)\,u=0$ in $(0,\infty)\times\Omega(x,\sqrt{s})$,

$$\sup_{y\in\Omega(x,c\sqrt{s})}\left|\left(\frac{\partial}{\partial t}\right)^m u(st_1,y)\right|\leqslant Cs^{-m}\inf_{y\in\Omega(x,c\sqrt{s})}u(st_2,y), \tag{73}$$

cf. [24, Theorem V.4.1]. The above estimates imply

$$|t^m h_t^m(x)| \leqslant Ct^{-n/2}, \qquad \text{for} \quad |x| \leqslant \sqrt{t}, \tag{74}$$

$$|t^m h_t^m(x)| \leqslant Ct^{-n/2} e^{-(|x|/\sqrt{t})^2}, \qquad \text{for} \quad \sqrt{t} \leqslant |x| \leqslant 1, \tag{75}$$

$$|t^m h_t^m(x)| \leqslant Ch(x) \qquad \text{for} \quad |x| > 1, \quad h \in L_1(G). \tag{76}$$

Thus using Lemma 3 we can prove the inequality (71). Since the estimates from the fourth step of the proof of Theorem 2 are still valid, one can prove the opposite inequality as above using the formula (64).

To prove the theorem for $s \leqslant 0$ one can use the Schwartz spaces introduce in [11] instead of the spaces $\mathscr{C}'(X)$. The proof for the Poisson semigroup is similar to the proof of Theorem 3. ▮

Remark 2. 1. For Besov spaces all above theorems remain true also for $0 < q < 1$. There are some technical difficulties for $F^s_{p,q}(X)$ with $q < 1$ but the theorems should hold also in this case.

2. The analogous theorem for function spaces on $\mathbb{R}^n$ can be found in [22]. In that case one has assumptions $m > s/2$ ($m > \max(s, 0)$ in the Poisson case). This is weaker that our assumption $m > |s|/2$. The difference comes from the method of proof. Namely from the duality argument use for $s < 0$.

3. Some partial result in this direction were presented in [12].

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